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Kinetic Limit of a Conservative Lattice Gas Dynamics Showing Long-Range Correlations

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An anisotropic lattice gas dynamics is investigated for which particles on $\mathbb{Z}^d$ jump to empty nearest neighbor sites with (fast) rate ε^{-2} in a specified direction and some particular configuration-dependent rates in the other directions. The model is translation and reflection invariant and is particle conserving. The space coordinate in the "fast-rate" direction is rescaled by ε^{-1}. It follows that the density field converges in probability, as $\varepsilon \downarrow 0$, to the corresponding solution of a nonlinear diffusion-type equation. The microscopic fluctuations about the deterministic macroscopic evolution are determined explicitly and it is found that the stationary fluctuations decay via a power law ($\sim 1/r^d$) with the direction dependence of a quadrupole field.

KEY WORDS: Stochastic lattice gases; kinetic limit; power law decay; fast rate exclusion process; long-range correlations.

1. INTRODUCTION

Recently, it was observed that homogeneous lattice gases subject to certain "anisotropic nonequilibrium dynamics" may show long-range stationary correlations.[1,2] This is based on formal expansions around an infinite-temperature dynamics and is supported by extensive computer simulations. We start by describing one of the simplest models considered. It is a symmetric exclusion process on the square lattice $\mathbb{Z}^2$ in which the rates are chosen to be direction dependent. More precisely, at each site $i \in \mathbb{Z}^2$ there is an occupation variable $\eta(i)$ with value 0 if the site i is empty, 1 if it is occupied. The full particle configuration is denoted by $\eta = \{\eta(i), i \in \mathbb{Z}^2\} \in \Omega \equiv \{0, 1\}^{\mathbb{Z}^2}$. The configurations η evolve according to a stochastic hopping

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dynamics. If the configuration before a jump is η, with $\eta(i)=1$, $\eta(j)=0$, then after the particle jumps from i to j, the configuration is η^{ij} with

$$\eta^{ij}(u)=\begin{cases}\eta(i) & \text{if } u=j\\ \eta(j) & \text{if } u=i\\ \eta(u) & \text{else}\end{cases} \tag{1.1}$$

The rate at which this jump occurs is denoted by $c(i, j, \eta)$. We choose them symmetrically (so that they can be interpreted as the rates at which the occupations at i and j are exchanged) and of the form

$$c(i, j, \eta)=c(j, i, \eta)=\begin{cases}1 & \text{if } i-j=\pm e_2\\ \Phi(\beta[H(\eta^{ij})-H(\eta)]) & \text{if } i-j=\pm e_1\\ 0 & \text{else}\end{cases} \tag{1.2}$$

where e_1 and e_2 are, respectively, the horizontal and vertical unit vectors in $\mathbb{Z}^2$. The energy $H(\eta)$ is the usual nearest neighbor Ising Hamiltonian with horizontal (K_1) and vertical (K_2) coupling:

$$H(\eta)=-4K_1 \sum_{i-j=\pm e_1} \eta(i)\,\eta(j)-4K_2 \sum_{i-j=\pm e_2} \eta(i)\,\eta(j) \tag{1.3}$$

The function Φ in (1.2) is normalized to $\Phi(0)=1$ and satisfies the relation

$$\Phi(z)=e^{-z}\Phi(-z), \qquad z\in\mathbb{R} \tag{1.4}$$

This choice implies that, at least for each of the lattice directions separately, the condition of detailed balance is satisfied, i.e.,

$$c_\alpha(i, j, \eta)=c_\alpha(i, j, \eta^{ij})\exp\{-\beta_\alpha[H(\eta^{ij})-H(\eta)]\}, \qquad \alpha=1, 2 \tag{1.5}$$

with $\beta_1\equiv\beta$, $\beta_2\equiv 0$, and $c_\alpha(i, j, \eta)=\Phi(\beta_\alpha[H(\eta^{ij})-H(\eta)])$ if $i-j=\pm e_\alpha$ (and is zero otherwise).

In the usual way (see, for example, refs. 3 and 4), the description above allows one to define a symmetric exclusion process η_t, $t>0$, with *speed change* given by (1.2). The *generator* L of this Markov process can be written as

$$L=L_{0,2}+L_{\beta,1} \tag{1.6}$$

where $L_{\beta_\alpha, e_\alpha}$, $\alpha=1, 2$, generates the exchanges of the occupation variables in the e_α direction at inverse temperature β_α, that is, for any local function f on Ω,

$$L_{\beta_\alpha, e_\alpha} f(\eta)\equiv\sum_{\langle ij\rangle} c_\alpha(i, j, \eta)[f(\eta^{ij})-f(\eta)] \tag{1.7}$$

with the sum over all nearest neighbor bonds $\langle ij\rangle$.

We are interested in a characterization of the stationary states for this process, and, in particular, in the behavior of the correlations in such a state for small but nonzero β (high temperature). At $\beta = 0$, there is a complete description of the stationary states[3]; they are the convex combinations of product measures with constant density ρ, $0 \leqslant \rho \leqslant 1$. It is argued in refs. 1 and 2 that putting $\beta \neq 0$, β small, has a rather drastic effect: the stationary covariance would only decay as a power and with a strong direction dependence. The Fourier transform $\hat{S}(k)$ of the stationary covariance $\mathbb{E}[(\eta(0) - \rho)(\eta(i) - \rho)] \equiv C(i)$ was calculated to first order in β. There is a constant $Q \sim \beta$ such that

$$\hat{S}(k) \approx \text{const} + Q \frac{k_1^2 - k_2^2}{k^2} \tag{1.8}$$

for small k. It implies a decay

$$C(i) \approx Q \frac{x^2 - y^2}{(x^2 + y^2)^2} \qquad \text{for} \quad i \equiv (x, y) \to \infty \tag{1.9}$$

typical of an electrostatic potential generated by a charge distribution with quadrupole moment Q. Note that such behavior is unheard of in equilibrium statistical mechanics, where, at high temperature, the truncated correlations decay in the same sense as the potential.[5] In particular, if we impose reversibility for the dynamics with respect to the Gibbs measures ν_H for interaction H of (1.3) by putting $\beta_1 = \beta_2 \equiv \beta$ in (1.5), then (1.9) has to be replaced by the first-order term in a high-temperature expansion for the equilibrium covariance $C_H(i)$ which is nonzero only if $i = 0$ or $i = \pm e_\alpha$, $\alpha = 1, 2$, i.e., $C_H(i) = O(\beta^2)$ if $|i| > 1$. We refer to ref. 2 for a detailed description and discussion of this nonequilibrium phenomenon, including further references.

While a rigorous analysis of this effect for the moment seems rather hard, the goal of this paper is to study this phenomenon exactly in the limit where the exchanges in the vertical direction become very fast. That is, the new rate in (1.2) is $c(i, j, \eta) = \varepsilon^{-2}$ if $i - j = \pm e_2$ with $\varepsilon \downarrow 0$. At the same time, the vertical space coordinate will be rescaled by ε^{-1}. Direct information about the stationary states for the microscopic model (1.6) is lost in this procedure, but we will recover the effect in the study of the fluctuations of the appropriately rescaled density field. A similar study was done by van Beijeren[6] for a simple diffusion model, where, using semi-phenomenological arguments, the appropriate fluctuating hydrodynamics is shown to give the same type of long-range correlations as we are finding here. A unified treatment on the microscopic level of these and other models using weak coupling expansions is in progress. Finally, let us emphasize that the model is translation invariant, in contrast with previously considered systems (see, for example, refs. 7 and 8).

A second motivation to study the "fast rate limit" of (1.2) is to derive the macroscopic equation of fluctuating hydrodynamics generalizing and making rigorous a microscopic model of the type considered by van Beijeren and Schulman[9] and also investigated by Krug *et al.*[10] We have to add, however, that the space rescaling we will apply is only partial, keeping in this way to the middle between microscopic models and the fully macroscopic approach.

The physical ideas behind the study of the system in the fast rate limit are well understood[11]: for $\varepsilon \downarrow 0$ the system has two well-defined time scales: (i) a microscopic one in which there are no exchanges of particles between vertical columns and the system reaches a stationary state within each column corresponding to the infinite-temperature dynamics in the vertical direction, and (ii) a macroscopic scale in which the system has exchanges of particles between columns with rates which are configuration dependent. We refer to the above and refs. 12–14 for the physical context and origin of related models.

The mathematical techniques we will use, are very similar to those employed in the derivation of reaction-diffusion equations for interacting particle systems (see, for example, ref. 15) and we consider our treatment here as an application of by now well-established methods[16] to the rigorous study (the first, to our knowledge) of an interesting nonequilibrium phenomenon: the origin of long-range spatial correlations.

In the next section we describe in detail the *kinetic* limit we are considering. In this limit the trajectories of the Markov process become concentrated on the solution of a nonlinear diffusion equation. Section 3 contains the results on the microscopic fluctuations about the deterministic equation. The appropriately rescaled fluctuation process determines in the limit a generalized Ornstein–Uhlenbeck process. In particular, the fluctuations about constant densities (= the only stationary and homogeneous solutions to the evolution equation at high temperature) are analyzed and it is found that the spectral density of the limiting process is nonanalytic at the origin in Fourier space. It implies a weak decay of the corresponding covariance. Section 4 is devoted to the proofs of these results.

2. THE FAST RATE LIMIT

The following analysis will be carried out in two space dimensions, but the higher-dimensional case poses—except for notation—no additional difficulties. We therefore continue to use the notation of the previous section and we write $i = xe_1 + ye_2 = (x, y) \in \mathbb{Z}^2$ for a general site.

Let $\varepsilon > 0$ be small. By speeding up the exchanges in the vertical direction we mean to replace (1.6) by the generator

$$L^\varepsilon = \varepsilon^{-2} L_{0,2} + L_{\beta,1} \tag{2.1}$$

That is, the rate for exchanges in the vertical direction is now ε^{-2}. In order to preserve spatial variations of the density field in the vertical direction, we have to rescale the corresponding space coordinate by ε^{-1}. The initial data are distributed according to a product measure μ^ε, with densities

$$\mu^\varepsilon(\eta(x, y)) = \rho_0(x, \varepsilon y) \tag{2.2}$$

where $\rho_0(x, r)$, $x \in \mathbb{Z}$, is smooth in $r \in \mathbb{R}$ and $0 \leqslant \rho_0(x, r) \leqslant 1$. Let η_t^ε, $t > 0$, be the process with generator L^ε of (2.1) and let $\mathbb{E}_{\mu^\varepsilon}^\varepsilon$ denote the corresponding expectation. We will argue that in the limit $\varepsilon \downarrow 0$, $\mathbb{E}_{\mu^\varepsilon}^\varepsilon(\eta_t^\varepsilon(x, y))$ is *close* to $\rho_t(x, \varepsilon y)$, a solution to the following nonlinear diffusion-type equation:

$$\frac{\partial}{\partial t} \rho_t(x, r) = \frac{\partial^2}{\partial r^2} \rho_t(x, r) - \nabla_x^* J_x(\rho_t(\cdot, r)) \tag{2.3}$$

where the last term in (2.3) is the discrete divergence $[\nabla_x^* f(x) \equiv f(x-1) - f(x)]$ of the current $J_x(\rho_t(\cdot, r))$ between (x, r) and $(x+1, r)$ which, itself, is a difference of the particle transport per unit time from (x, r) to $(x+1, r)$ and back:

$$\begin{aligned} J_x(\rho_t(\cdot, r)) = {} & R(\rho_t(x+2, r), \rho_t(x+1, r), \rho_t(x, r), \rho_t(x-1, r)) \\ & - R(\rho_t(x-1, r), \rho_t(x, r), \rho_t(x+1, r), \rho_t(x+2, r)) \end{aligned} \tag{2.4}$$

$R(\cdot)$ is the expected rate of a particle jumping to a nearest neighbor site in the horizontal direction,

$$\begin{aligned} & R(\rho^{(1)}, \rho^{(2)}, \rho^{(3)}, \rho^{(4)}) \\ & \quad \equiv \langle \Phi(-4\beta[K_1 \eta(2, 2)\, \eta(2, 2)\, \eta(4, 2) + K_1 \eta(3, 2)\, \eta(1, 2) \\ & \qquad + K_2 \eta(2, 2)\, \eta(3, 3) + K_2 \eta(3, 2)\, \eta(2, 3) + K_2 \eta(3, 2)\, \eta(2, 1) \\ & \qquad + K_2 \eta(2, 2)\, \eta(3, 1) - K_1 \eta(3, 2)\, \eta(4, 2) - K_1 \eta(2, 2)\, \eta(1, 2) \\ & \qquad - K_2 \eta(3, 2)\, \eta(3, 3) - K_2 \eta(3, 2)\, \eta(3, 1) - K_2 \eta(2, 2)\, \eta(2, 3) \\ & \qquad - K_2 \eta(2, 2)\, \eta(2, 1)])\, \eta(2, 2)[1 - \eta(3, 2)] \rangle_{\rho^{(1)}, \ldots, \rho^{(4)}} \end{aligned} \tag{2.5}$$

with respect to the product measure having densities

$$\langle \eta(k, l) \rangle_{\rho^{(1)}, \ldots, \rho^{(4)}} \equiv \rho^{(k)}, \qquad k = 1, 2, 3, 4; \quad l = 1, 2, 3 \tag{2.6}$$

The initial condition to Eq. (2.3) is given by ρ_0 [see (2.2)]. Note that the current (2.4) is a polynomial in the density and therefore, using iteration, one can prove

Proposition 1. For any initial condition $\rho_0(x, r)$ smooth in $r \in \mathbb{R}$, $0 \leqslant \rho_0 \leqslant 1$, with uniformly bounded derivatives, there is a unique solution $\rho_t(x, r)$ smooth in r, $0 \leqslant \rho_t \leqslant 1$, to (2.3), having uniformly bounded derivatives in r.

Note that the dynamics (1.2) was effectively changed by putting in the factor ε^{-2} in (2.1). Therefore, the limit $\varepsilon \downarrow 0$ is called a *kinetic* limit,[4] to distinguish it from other limiting procedures, such as the hydrodynamic limit. Equation (2.3) is a deterministic conservation equation for the time evolution of the density field, obtained in much the same way as reaction-diffusion equations.[15] The fast environment-independent exchanges in the vertical direction, combined with the vertical space rescaling, produce the linear diffusion term. The exchanges in the horizontal direction occur on a much slower time scale and their influence on the change in the density must be calculated in the so-called "local equilibrium measure," i.e., the distribution at time t in a box of vertical size ε^{-1}, looks like a product measure with instantaneous densities which vary column per column in the horizontal direction, which is kept discrete. Here, Eq. (2.3) appears only as a first step in the study (in the next section) of the fluctuations. Its derivation involves proving a *law of large numbers*, the mathematical mechanism underlying the ideas above. We present without proof one of the more relevant results in that respect.

Proposition 2. For all $T \geqslant 0$ and $n \geqslant 0$,

$$\lim_{\varepsilon \to 0} \sup_{0 \leqslant t \leqslant T} \sup_{\substack{A \subset \mathbb{Z}^2 \\ |A| = n}} \left| \mathbb{E}^{\varepsilon}_{\mu^\varepsilon} \left[\prod_{i \in A} \eta_t(i) \right] - \prod_{\substack{i \in A \\ i = (x, y)}} \rho_t(x, \varepsilon y) \right| = 0 \tag{2.7}$$

where $\rho_t(x, r)$ solves (2.3). Here $|A|$ is the cardinality of the set A.

3. THE FLUCTUATION PROCESS

We want to determine the microscopic fluctuations about the deterministic evolution equation (2.3). Formally (and following the notation of ref. 10), the limiting process describing the density fluctuation field will be given, in some generalized weak form, by a linear Langevin equation:

$$\frac{\partial}{\partial t} \xi_t(x, r) = \frac{\partial^2}{\partial r^2} \xi_t(x, r) - \nabla_x^* \mathscr{I}_x(\xi_t(\cdot, r)) + W(x, r, t) \tag{3.1}$$

with linearized horizontal current

$$\begin{aligned}\mathscr{I}_x(\xi_t(\cdot, r)) = {} & R_{14}(\rho_t(x+2, r), \rho_t(x+1, r), \rho_t(x, r), \rho_t(x-1, r)) \\ & \times [\xi_t(x+2, r) - \xi_t(x-1, r)] \\ & + R_{23}(\rho_t(x+2, r), \rho_t(x+1, r), \rho_t(x, r), \rho_t(x-1, r)) \\ & \times [\xi_t(x+1, r) - \xi_t(x, r)] \end{aligned} \tag{3.2}$$

$$R_{kl} \equiv \left[\frac{\partial}{\partial \rho^{(k)}} - \frac{\partial}{\partial \rho^{(l)}}\right] R, \qquad k, l = 1, \dots, 4 \tag{3.3}$$

[the function R was defined in (2.5)] and where $W(x, r, t)$ in (3.1) is a "white noise" with covariance

$$\begin{aligned}&\langle W(x, r, t)\, W(x', r', t')\rangle \\ &\quad = \Bigg(\delta_{xx'} \frac{\partial^2}{\partial r\, \partial r'} \{\rho_t(x, r)[1 - \rho_t(x, r)]\, \delta(r - r')\} \\ &\qquad + \delta(r - r')\, \nabla_x^* \nabla_{x'}^* [a_x(\rho_t(\cdot, r))\, \delta_{xx'}]\Bigg)\, \delta(t - t') \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}a_x(\rho_t(\cdot, r)) \equiv {} & R(\rho_t(x+2, r), \rho_t(x+1, r), \rho_t(x, r), \rho_t(x-1, r)) \\ & + R(\rho_t(x-1, r), \rho_t(x, r), \rho_t(x+1, r), \rho_t(x+2, r)) \end{aligned} \tag{3.5}$$

More precisely, for all $\varepsilon > 0$ and $A \in \mathscr{A}$ ($\mathscr{A} \equiv$ the finite subsets of $\mathbb{Z}^2$), $\phi \in \mathscr{S}$ ($\mathscr{S}$ is the Schwartz space of rapidly decreasing functions), we define (now using the notation of ref. 15) the random variables (or the *fluctuation field*)

$$Y_t^\varepsilon(A, \phi) \equiv \sqrt{\varepsilon} \sum_{x \in A} \sum_{y \in \mathbb{Z}} \phi(\varepsilon y)[\eta_t^\varepsilon(x, y) - \mathbb{E}_{\mu^\varepsilon}^\varepsilon(\eta_t(x, y))] \tag{3.6}$$

For $A = \{x\}$, we write $Y_t^\varepsilon(A, \phi) = Y_t^\varepsilon(x, \phi)$. One must consider $\{Y_t^\varepsilon\}$ as a process on the path space $D([0, \infty), \mathscr{L})$ with state space $\mathscr{L}$ consisting of the functionals G on $\mathscr{A} \times \mathscr{S}$ which are additive on $\mathscr{A}$ ($G(A \cup B. \phi) = G(A, \phi) + G(B, \phi)$ if $A \cap B = \varnothing$] and linear on $\mathscr{S}$ (Schwartz distributions). Define for $Y \in \mathscr{L}$

$$\begin{aligned}\mathscr{I}_x(Y, \phi) \equiv {} & Y(x+2, V_x^{14}\phi) - Y(x-1, V_x^{14}\phi) \\ & + Y(x+1, V_x^{23}\phi) - Y(x, V_x^{23}\phi) \end{aligned} \tag{3.7a}$$

with $V_x^{kl}\phi \in \mathscr{S}$, $k, l = 1, \dots, 4$, given by

$$V_x^{kl}\phi(r) = R_{kl}(\rho_t(x+2, r), \rho_t(x+1, r), \rho_t(x, r), \rho_t(x-1, r))\, \phi(r) \tag{3.7b}$$

Finally, put

$$\|B_s(A, \phi)\|^2 = \sum_{x, x' \in A} \langle B_s(x, \phi)\, B_s(x', \phi)\rangle \tag{3.8}$$

with

$$\begin{aligned} &\langle B_s(x, \phi)\, B_s(x', \psi)\rangle \\ &\quad \equiv \int dr\, \phi'(r)\, \psi'(r)\, \delta_{x,x'}\, \rho_s(x, r)[1 - \rho_s(x, r)] \\ &\qquad + \int dr\, \phi(r)\, \psi(r) [\delta_{x,x'} [a_x(\rho_s(\cdot, r)) + a_{x-1}(\rho_s(\cdot, r))] \\ &\qquad - \delta_{x,x'+1}\, a_{x-1}(\rho_s(\cdot, r)) - \delta_{x,x'-1}\, a_x(\rho_s(\cdot, r))] \end{aligned} \tag{3.9}$$

Let P^ε be the probability distribution of $\{Y_t^\varepsilon\}$.

Proposition 3. $P^\varepsilon \to P$ weakly as $\varepsilon \downarrow 0$, where P is the distribution of a generalized Ornstein–Uhlenbeck process $\{Y_t\}$, uniquely determined by the condition that the $Y_t(A, \phi)$ are centered, that for all $g \in C_0^\infty$,

$$\begin{aligned} &g(Y_t(A, \phi)) - \int_0^t ds\, g'(Y_s(A, \phi)) [Y_s(A, \phi'') - \sum_{x \in A} \nabla_x^* \mathscr{I}_x(Y_s, \phi))] \\ &\qquad - \int_0^t ds\, g''(Y_s(A, \phi))\, \|B_s(A, \phi)\|^2 \end{aligned} \tag{3.10}$$

is a P-martingale, and that the law of the initial condition $\{Y_0(x, \phi)\}$ is Gaussian with covariance

$$P(Y_0(A, \phi)\, Y_0(B, \psi)) = \delta_{A,B} \sum_{x \in A} \int dr\, \phi(r)\, \psi(r)\, \rho_0(x, r)[1 - \rho_0(x, r)] \tag{3.11}$$

Consider now the constant solutions $\rho_t(x, r) \equiv \rho$, $0 \leqslant \rho \leqslant 1$, of Eq. (2.3). For that choice, the currents $\{J_x\}$ vanish identically. The equal-time correlations of the fluctuation field are now most easily described by Fourier transforming the formal stochastic differential equation (3.1). We get

$$\begin{aligned} \frac{\partial}{\partial t} \hat{\xi}_t(k_1, k_2) &= \hat{W}(k_1, k_2, t) - k_2^2 \hat{\xi}_t(k_1, k_2) - 2\hat{\xi}_t(k_1, k_2) \\ &\qquad \times \{R_{23}(\rho) + [R_{14}(\rho) - R_{23}(\rho)] \cos k_1 - R_{14}(\rho) \cos 2k_1\} \end{aligned} \tag{3.12}$$

where

$$\langle \hat{W}(k_1, k_2, t)\, \hat{W}(k_1', k_2', t')\rangle$$
$$\equiv \delta(k-k')\,\delta(t-t')[k_2^2\rho(1-\rho)+2R(\rho)(1-\cos k_1)] \quad (3.13)$$

with $k=(k_1, k_2)$ in the strip $-\pi \leqslant k_1 < \pi$, $-\infty < k_2 < \infty$, and we have used in (3.11) and (3.12), $R(\rho) \equiv R(\rho, \rho, \rho, \rho)$, etc., as short-hand notations. The static *structure function*, i.e., the Fourier transform of the stationary covariance for the process described in (3.12), is now easy to compute:

$$\hat{S}(k_1, k_2) = \frac{k_2^2\rho(1-\rho)+2R(\rho)(1-\cos k_1)}{k_2^2+2[R_{23}(\rho)+R_{14}(\rho)(1+2\cos k_1)](1-\cos k_1)} \quad (3.14)$$

Note that while $\hat{S}(k)$ remains bounded at $k=0$, it is not analytic whenever

$$F(\rho) \equiv R(\rho) - \rho(1-\rho)[R_{23}(\rho)+3R_{14}(\rho)] \neq 0 \quad (3.15)$$

Of course, $F(\rho)=0$ at $\beta=0$, for which $R(\rho)=\Phi(0)\,\rho(1-\rho)$, $R_{23}(\rho)=\Phi(0)$, and $R_{14}(\rho)=0$. However, before further investigating (3.14) and condition (3.15), we give a more precise formulation of the discussion above.

Proposition 4. The equal-time covariance of the limiting process $\{Y_t\}$, described in Proposition 3, satisfies

$$\begin{aligned} &P(Y_t(A, \phi)\, Y_t(B, \psi)) \\ &\quad = P(Y_0(A, \phi)\, Y_0(B, \psi)) \\ &\qquad + P\left\{\int_0^t ds \left[Y_s(A, \phi'') - \sum_{x \in A} \nabla_x^* \mathscr{I}_x(Y_s, \phi))\right] Y_s(B, \psi)\right. \\ &\qquad \left. + \int_0^t ds\, Y_s(A, \phi) \left[Y_s(B, \psi'') - \sum_{x' \in B} \nabla_{x'}^* \mathscr{I}_{x'}(Y_s, \psi))\right]\right\} \\ &\qquad + \int_0^t ds\, \langle B_s(A, \phi)\, B_s(B, \psi)\rangle \end{aligned} \quad (3.16)$$

It follows that the stationary covariance of the liiting process $\{Y_t\}$ corresponding to the constant solutions $\rho_t(x, r) \equiv \rho$, $0 \leqslant \rho \leqslant 1$, of Eq. (2.3) is given by

$$\begin{aligned} &P(Y_\infty(x, \phi)\, Y_\infty(x', \psi)) \\ &\quad = \int_{-\infty}^{\infty} dk_2 \int_{-\pi}^{\pi} dk_1\, \hat{\phi}(k_2)\, \hat{\psi}(k_2)\, \hat{S}(k_1, k_2) e^{ik_1 \cdot (x-x')} \end{aligned} \quad (3.17)$$

where the caret denotes Fourier transform and $\hat{S}$ is the spectral density given in (3.14).

The stationary covariance kernel corresponding to (3.17) has a decay

$$S(x, r) \approx F(\rho) \frac{x^2 - cr^2}{(x^2 + cr^2)^2} \tag{3.18}$$

as $|(x, r)|^2 = x^2 + r^2 \to \infty$, for some constant c. We thus recover the phenomenon of power law decay of the static covariance, as described in ref. 2.

It remains to investigate condition (3.15) to find more explicit situations under which this nonequilibrium effect occurs. $F(\rho)$ also depends on the function Φ and the coupling parameters $a \equiv -4\beta K_1$ and $b \equiv -4\beta K_2$, and can be calculated starting from (2.5)

$$\begin{aligned} F(\rho) = \rho^2(1-\rho)^2 \, (2[&\Phi(b) - \Phi(-b)][\rho^2 + (1-\rho)^2]^2 \\ &+ 3[\Phi(a) - \Phi(-a)]\{[\rho^2 + (1-\rho)^2]^2 + 2\rho^2(1-\rho)^2\} \\ &+ [8\Phi(a+b) - 8\Phi(-a-b) + 4\Phi(a-b) - 4\Phi(-a+b) \\ &+ 2\Phi(2b) - 2\Phi(-2b)] \, \rho(1-\rho)[\rho^2 + (1-\rho)^2] + [5\Phi(a+2b) \\ &- 5\Phi(-a-2b) + \Phi(a-2b) - \Phi(-a+2b)] \, \rho^2(1-\rho)^2) \end{aligned} \tag{3.19}$$

This expression is general and the following special cases can be considered. If $b = 0$ [no vertical coupling in the Hamiltonian (4.2)], then

$$F(\rho) = 3\rho^2(1-\rho)^2 [\Phi(a) - \Phi(-a)] \tag{3.20}$$

which is never zero unless $\rho = 0$ or $\rho = 1$, or $\Phi(a) = \Phi(-a)$ corresponding to an infinite-temperature dynamics; if $K_1 \geqslant 2K_2 > 0$ (ferromagnetic case with large enough horizontal coupling), then using the extra (detailed balance) condition $\Phi(z) = e^{-z}\Phi(-z)$, it can be checked that all terms in (3.19) are strictly positive whenever $\rho \neq 0$, $1 - \rho \neq 0$, implying again power law decay of the static pair correlations. Of course the same is true for $K_1 \leqslant 2K_2 < 0$.

4. PROOFS

Proof of Proposition 3. We sketch here the main ideas in the proof of Proposition 3. The details are almost identical to the arguments presented for reaction-diffusion processes in ref. 15. The general framework consists of the Holley–Stroock[17] martingale approach to the problem of tightness of the family $\{P^\varepsilon\}$ and to the characterization of a limiting point P. The input is made up of some *a priori* bounds on the equal-time correlation functions and their behavior as $\varepsilon \downarrow 0$. It is there that the main reason appears why the fast rate limit of our model and the fluctuations about this limit are exactly solvable. To zeroth order in ε, the distribution at time t is an independent collection of one-dimensional product measures. It is only at first order in ε that the model becomes two dimensional with the creation of correlations between different (vertical) columns. The underlying dynamics on which the whole problem rests is the simple symmetric exclusion process, whose behavior is well understood (see, for example, refs. 16 and 18). We now give a more explicit exposition of these various aspects and how they appear in the proof.

Define for every $A \in \mathscr{A}$, $\phi \in \mathscr{S}$,

$$\gamma_1^\varepsilon(A, \phi; t) \equiv L^\varepsilon Y_t^\varepsilon(A, \phi) - \mathbb{E}_{\mu^\varepsilon}^\varepsilon[L^\varepsilon Y_t^\varepsilon(A, \phi)] \tag{4.1a}$$

$$\gamma_2^\varepsilon(A, \phi; t) \equiv L^\varepsilon[Y_t^\varepsilon(A, \phi)]^2 - 2Y_t^\varepsilon(A, \phi)\, L^\varepsilon Y_t^\varepsilon(A, \phi) \tag{4.1b}$$

For every $t \geqslant 0$,

$$M^\varepsilon(t) \equiv Y_t^\varepsilon(A, \phi) - \int ds\, \gamma_1^\varepsilon(A, \phi; s) \tag{4.2a}$$

$$N^\varepsilon(t) \equiv [M^\varepsilon(t)]^2 - \int ds\, \gamma_2^\varepsilon(A, \phi; s) \tag{4.2b}$$

are P^ε-martingales with respect to the canonical filtration $\{\mathscr{M}_t\}$. To show that for any $T > 0$, P^ε is tight in $[0, T]$ and that any limiting point has support on $C([0, \infty), \mathscr{L})$, it is sufficient[17,19-21] to obtain the following bounds: for every $A \in \mathscr{A}$, $\phi \in \mathscr{S}$,

$$\sup_{0 \leqslant t \leqslant T} \mathbb{E}_{\mu^\varepsilon}^\varepsilon[(Y_t^\varepsilon(A, \phi))^2] < \infty \tag{4.3}$$

and

$$\sup_{\varepsilon} \sup_{0 \leqslant t \leqslant T} \mathbb{E}_{\mu^\varepsilon}^\varepsilon[(\gamma_v^\varepsilon(A, \phi; t))^2] < \infty, \qquad v = 1, 2 \tag{4.4}$$

To prove the uniqueness of a limiting point, we have to show that if P^ε converges weakly to P, then the expression (3.7) is a P-martingale. This will follow from the fact that, for every $g \in C_0^\infty(\mathbb{R})$,

$$g(Y_t^\varepsilon(A, \phi)) - \int_0^t ds\, \{L^\varepsilon g(Y_s^\varepsilon(A, \phi)) - \mathbb{E}_{\mu^\varepsilon}^\varepsilon[L^\varepsilon g(Y_s^\varepsilon(A, \phi))]\} \tag{4.5}$$

is a P^{ε}-martingale with

$$\begin{aligned} & L^{\varepsilon} g(Y_s^{\varepsilon}(A, \phi)) - \mathbb{E}_{\mu^{\varepsilon}}^{\varepsilon}[L^{\varepsilon} g(Y_s^{\varepsilon}(A, \phi))] \\ &\quad = \gamma_1^{\varepsilon}(A, \phi; s)\, g'(Y_s^{\varepsilon}(A, \phi)) \\ &\qquad + \tfrac{1}{2}\gamma_2^{\varepsilon}(A, \phi; s)\, g''(Y_s^{\varepsilon}(A, \phi)) + o(\varepsilon) \end{aligned} \tag{4.6}$$

$\{\lim_{\varepsilon \to 0}[o(\varepsilon)/\varepsilon] = 0\}$ and γ_1^{ε}, γ_2^{ε} "converge" to the corresponding term in (3.9), i.e.,

$$\lim_{\varepsilon \to 0} \mathbb{E}_{\mu^{\varepsilon}}^{\varepsilon}[|\gamma_2^{\varepsilon}(A, \phi; t) - \|B_t(A, \phi)\|^2|] = 0 \tag{4.7a}$$

$$\begin{aligned} \lim_{\varepsilon \to 0} \mathbb{E}_{\mu^{\varepsilon}}^{\varepsilon}\Bigg[\Psi\Bigg(\int_{T_1}^{T_2} dt \Bigg\{ & - Y_t^{\varepsilon}(A, \phi'') + \sum_{x \in A} \nabla_x^* \mathscr{I}_x(Y_s^{\varepsilon}(\cdot, \phi)) \\ & + \gamma_1^{\varepsilon}(A, \phi; t) \Bigg\}\, g'(Y_t^{\varepsilon}(A, \phi)) \Bigg) \Bigg] = 0 \end{aligned} \tag{4.7b}$$

for every $0 \leqslant T_1 \leqslant T_2$ and bounded continuous function Ψ measurable with respect to $\mathscr{M}_{T_1}$, the σ-algebra generated by $\{Y_t^{\varepsilon}(A, \phi),\ 0 \leqslant t \leqslant T_1,\ A \in \mathscr{A},\ \phi \in \mathscr{S}\}$.

We now consider the various conditions (4.3)–(4.7) in more detail. Condition (4.3) is trivial given Proposition 2. In fact, the cross terms in $[Y_t^{\varepsilon}(A, \phi)]^2$ are of order ε and the diagonal terms give

$$\varepsilon \sum_{y \in \mathbb{Z}} \sum_{x \in A} [\phi(\varepsilon y)]^2\, \mathbb{E}_{\mu^{\varepsilon}}^{\varepsilon}[\eta_t(x, y)]\{1 - \mathbb{E}_{\mu^{\varepsilon}}^{\varepsilon}[\eta_t(x, y)]\} \tag{4.8}$$

which is close to

$$\int dr \sum_{x \in A} [\phi(r)]^2\, \rho_t(x, r)[1 - \rho_t(x, r)] < \infty$$

To verify (4.4)–(4.7), we need a more explicit form of γ_1^{ε}, γ_2^{ε} [defined in (4.1)]:

$$\gamma_1^{\varepsilon}(A, \phi; t) = Y_t^{\varepsilon}(A, \Delta^{(\varepsilon)}\phi) - \sum_{x \in A} \nabla_x^* Y_t^{\varepsilon}(x, \phi; f) \tag{4.9a}$$

with

$$f(\eta) \equiv c_1((0, 0), (1, 0), \eta)[\eta(1, 0) - \eta(0, 0)] \tag{4.9b}$$

$[c_1(\cdot)$ is as in (1.5)], and

$$Y_t^{\varepsilon}(x, \phi; f) \equiv \sqrt{\varepsilon} \sum_{y \in \mathbb{Z}} \phi(\varepsilon y)[\tau_{-(x, y)} f(\eta_t^{\varepsilon}) - \mathbb{E}_{\mu^{\varepsilon}}^{\varepsilon}(\tau_{-(x, y)} f(\eta_t^{\varepsilon}))] \tag{4.9c}$$

$[\tau_{-(x,y)}$ is the shift over lattice vector $(x, y)]$ and

$$\Delta^{(\varepsilon)}\phi(r) \equiv \frac{1}{\varepsilon^2}\left[\phi(r+\varepsilon) + \phi(r-\varepsilon) - 2\phi(r)\right] \tag{4.9d}$$

$$\gamma_2^\varepsilon(A, \phi; t) = \frac{\varepsilon}{2}\sum_{x \in A}\sum_{y \in \mathbb{Z}}\left[\frac{\phi(\varepsilon y) - \phi(\varepsilon(y+1))}{\varepsilon}\right]^2 [\eta_t^\varepsilon(x, y) - \eta_t^\varepsilon(x, y+1)]^2$$
$$+ \frac{\varepsilon}{2}\sum_{x, x' \in A}\sum_{y \in \mathbb{Z}} [\phi(\varepsilon y)]^2 [\delta_{x,x'}(\tilde{a}_{x,y}(\eta_t^\varepsilon) + \tilde{a}_{x-1,y}(\eta_t^\varepsilon))$$
$$- \delta_{x,x'+1}\tilde{a}_{x-1,y}(\eta_t^\varepsilon) - \delta_{x,x'-1}\tilde{a}_{x,y}(\eta_t^\varepsilon)] \tag{4.10a}$$

with

$$\tilde{a}_{x,y}(\eta) \equiv \tau_{-(x,y)}\tilde{a}_0(\eta), \qquad \tilde{a}_0(\eta) \equiv c_1((0,0), (1,0), \eta)[\eta(0,0) - \eta(1,0)]^2 \tag{4.10b}$$

It is easy to see that (4.4) holds trivially for γ_2^ε. Condition (4.4) for γ_1^ε follows again from Proposition 2. Conditions (4.5) and (4.6) are well-known results that do not depend on our specific model. Condition (4.7a) is a consequence of Proposition 2. So only (4.7b) remains to be shown. Note that (4.7b) asserts that one can approximate γ_1^ε by linearizing around the solution to Eq. (2.3). It involves a rather deep property of the fluctuation field of a local function [see (4.9c)]: it is governed by the fluctuations of the density field. A precise formulation is given in Theorem 4 of ref. 15, which, in our case, states that for every $0 < \tau' < \tau < \infty$,

$$\lim_{T \to \infty}\lim_{\varepsilon \to 0}\sup_{\tau' \leqslant t \leqslant \tau} \mathbb{E}_{\mu^\varepsilon}^\varepsilon\left[\left|\frac{1}{\varepsilon^2 T}\int_t^{t+\varepsilon^2 T} ds\, [Y_s^\varepsilon(x, \phi; f) - Y_s^\varepsilon(x, b_s^f\phi)]\right|^2\right] = 0 \tag{4.11a}$$

where

$$b_s^f\phi(r) \equiv \phi(r)\sum_{x \in \mathbb{Z}} \frac{\partial}{\partial \rho^{(x)}}\nu_{\{\rho^{(x)}\}}(f)\bigg]_{\rho^{(x)} = \rho_s(x, r),\, x \in \mathbb{Z}} \tag{4.11b}$$

where $\nu_{\{\rho(x)\}}$ is a product measure on Ω with density

$$\nu_{\{\rho(x)\}}(\eta(x, y)) \equiv \rho^{(x)}, \qquad x, y \in \mathbb{Z} \tag{4.11c}$$

The proof of (4.11) can be copied from ref. 15 by using the appropriate (generalized) duality. This duality is obtained by writing the horizontal exchange rates as

$$c_1(i, j, \eta) = \sum \lambda_\alpha I[\eta \in A_\alpha(i, j)] \tag{4.12}$$

with $\lambda_\alpha \geqslant 0$ and $\{A_\alpha(i, j)\}_\alpha$, a finite partition of the configuration space Ω for which

$$c_1(i, j, \eta) = \lambda_\alpha \qquad \text{iff} \quad \eta \in A_\alpha(i, j) \tag{4.13}$$

The generator of the process can thus be written as

$$L^\varepsilon f(\eta) = \varepsilon^{-2} \sum_{i-j=\pm e_2} [f(\eta^{ij}) - f(\eta)] + \sum_{i-j=\pm e_1} \sum_\alpha \lambda_\alpha [f(\eta^{\alpha, ij}) - f(\eta)] \tag{4.14a}$$

where

$$\eta^{\alpha, ij} = \begin{cases} \eta & \text{if} \quad \eta \notin A_\alpha(i, j) \\ \eta^{ij} & \text{if} \quad \eta \in A_\alpha(i, j) \end{cases} \tag{4.14b}$$

To each horizontal bond of nearest neighbors on $\mathbb{Z}^2$, we associate a collection of independent Poisson processes with intensities λ_α and to each vertical bond there is associated a Poisson clock with intensity ε^{-2}. The dual process is now easy to construct and one is in a position to carry over the ideas of ref. 15 to finish the proof. ∎

Proof of Proposition 4. Equations (3.16), (3.17) are an immediate consequence of Proposition 3. To prove (3.18) it is sufficient to require [from (3.16)] that

$$\begin{aligned} P\{[Y_\infty(x, \phi'') - \nabla_x^* \mathscr{I}_x(Y_\infty, \phi))]\, Y_\infty(x', \psi) \\ + Y_\infty(x, \phi)[Y_\infty(x', \psi'') - \nabla_{x'}^* \mathscr{I}_{x'}(Y_\infty, \psi))]\} \\ = -\langle B_\infty(x, \phi)\, B_\infty(x', \psi)\rangle \end{aligned} \tag{4.15}$$

and to plug in the constant solution $\rho_t(x, r) \equiv \rho$, for which

$$\begin{aligned} &-\nabla_x^* \mathscr{I}_x(Y_\infty, \phi)) \\ &\quad = R_{14}(\rho)[Y_\infty(x+2, \phi) - Y_\infty(x-1, \phi) - Y_\infty(x+1, \phi) + Y_\infty(x-2, \phi)] \\ &\qquad + R_{23}(\rho)[Y_\infty(x+1, \phi) - 2Y_\infty(x, \phi) + Y_\infty(x-1, \phi)] \end{aligned} \tag{4.16}$$

and $a_x \equiv 2R(\rho)$. ∎

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